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On Supplements in Modules

by

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0. Introduction

In a recent paper, K. Varadarajan established a criterion under which supplements in projective modules are direct summands. The purpose of this note is to weaken Varadarajan's condition, and to extend the result to quasi-projective modules. Our criterion will be the Hopficity of the radical. Hopfian quasiprojective modules will be characterized and criteria relating the Hopficity of a module to that of its radical will be given. In addition, we will indicate alternate short proofs of Varadarajan's characterization of perfect and semi-perfect rings.

1. Supplements with Hopfian radical

Notation and terminology will follow Varadarajan [9] and Anderson-Fuller [1]. All modules will be unital left modules. A module M is said to be *Hopfian* if every surjective endomorphism of M is an automorphism.

LEMMA 1.1. *Let* $f: M \to M$ *be an epimorphism and let* $\mathrm{Ker} f \subseteq J(M)$. *If* $J(M)$ *is Hopfian then* f *is an automorphism.*

Proof. By ([1]; p. 121, 9.15]), $J(M)=f(J(M))$. Since $J(M)$ is Hopfian, $0=\mathrm{Ker} f \cap J(M)=\mathrm{Ker} f$.

COROLLARY 1.2. *Suppose* $J(M)$ *is Hopfian. Then* M *is Hopfian if and only if every submodule* N *of* M *with* $M/N \simeq M$ *is small in* M.

Proof ([1]; p. 120, 9.13). Let N and H be submodules of M, and let $M=N+H$. If N and H are supplements of each other and M is quasi-projective, then $M=N\oplus H$ ([7]; p. 92, 2.3). However, even in a free module a given submodule need not have a supplement ([4]; Theorem 4.17); also, if H is a supplement, the existence of N so that H and N are mutual supplements is not certain; in particular, we do not know whether a supplement in a quasi-projective module is quasi-projective.

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The following result contains a criterion under which supplements in quasi-projective modules are direct summands and, thus, are quasi-projective ([11]; p. 442, 2.3).

THEOREM 1.3. *Let Q be a quasi-projective module, and let $H \subseteq Q$ be a supplement of some submodule of Q. If $J(H)$ is Hopfian then H is a direct summand of Q; in particular, H is quasi-projective.*

Remark. It is well known that every Noetherian module is Hopfian; also, finitely generated modules over commutative rings are Hopfian ([10]; p. 506, 1.2). Thus, Theorem 1.3 generalizes ([9]; p. 563, 2.4). That Hopfian modules in turn need not be finitely generated can be seen from the infinite Hopfian p-primary group constructed by R. S. Pierce ([8]; p. 302, 16.4).

Proof of Theorem 1.3. Let H be a supplement of $N \subseteq Q$. Then $Q = N + H$ and $N \cap H$ is small in H ([7]; p. 87, 1.3). By ([7]; p. 91, 2.1) there exists a homomorphism ψ: $Q \to H$ making the diagram

$$
\begin{tikzcd}
 & Q \arrow[dl, dashed, "\psi"'] \arrow[d, "\nu"] \\
H \arrow[r, "\alpha"'] & Q/N
\end{tikzcd}
$$

commutative, where ν is the natural epimorphism and $\alpha = \nu \mid H$. From $Q/N = \nu(H) = \alpha(\psi(H))$, it follows that $H = \psi(H) + H \cap N$, so that $H = \psi(H)$. Let $f = \psi \mid H$. Then $\operatorname{Ker} f \subseteq H \cap N \subseteq J(H)$, and f is an automorphism by Lemma 1.1. Since $\psi \circ f^{-1} = 1_H$, H is a direct summand.

2. Quasi-projective Hopfian modules

Theorem 1.3 motivates the consideration of quasi-projective modules with Hopfian radical. We have the following easy characterization of quasi-projective Hopfian modules.

PROPOSITION 2.1. *The quasi-projective module Q is Hopfian if and only if Q is not isomorphic to a proper direct summand of Q.*

Proof ([3]; p. 6, Lemma 4). If R is a ring and $J(R) = 0$ then $J(P) = 0$ for every projective R-module P ([1]; p. 196, 17.10). Thus $J(P)$ being Hopfian does not imply the Hopficity of P. At the other end of the scale, however, the situation is different:

PROPOSITION 2.2. *Let Q be a quasi-projective module such that $J(S) \neq 0$ for every non-zero direct summand S of Q. Then $J(Q)$ Hopfian implies Q Hopfian.*

Proof. Suppose $Q = C \oplus S$, $C \simeq Q$. Then $J(Q) = J(C) \oplus J(S)$ ([1]; p. 121, 9.19), and $J(C) \simeq J(Q)$. Since $J(Q)$ is Hopfian, $J(S) = 0$, so that $S = 0$.

COROLLARY 2.3. *Let Q be a quasi-projective module such that $J(S)\neq 0$ for every direct summand $S\neq 0$ of Q, and let $H\subseteq Q$ be a supplement in Q. If $J(H)$ is Hopfian, then H is a quasi-projective Hopfian direct summand of Q.*

Proof. Theorem 1.3 and Proposition 2.2.

3. Pefect and semi-perfect rings

The ring R is called left (semi-) perfect if every (finitely generated) left R-module has a projective cover ([2]; p. 471). While left semi-perfect rings are right semi-perfect, left perfect rings need not be right perfect.

For P a projective module, the following conditions are known to be equivalent ([6]; p. 526, Satz; and [7]; p. 96, 3.3): (i) P has property (P_1); (ii) every epimorphic image of P has a projective cover; (iii) P has property (P_2). Properties (P_1) and (P_2) are inherited by epimorphic images ([5]; p. 245, 11.1.4 (3); and [7]; p. 90, 1.12), and sums of finitely many modules satisfying (P_1) satisfy (P_1) ([5]; p. 268, Exercise 4b).

These remarks immediately prove Varadarajan's Theorems 1.6 and 1.7 ([9]; p. 561).

References

[1] ANDERSON, F. W. and FULLER, K. R.; "*Rings and Categories of Modules*," Springer-Verlag, New York, 1974.

[2] BASS, H.; Finitistic dimension and a homological generalization of semi-primary rings, *Trans. Amer. Math. Soc.*, **95** (1960), 466–488.

[3] FUCHS, L. and RANGASWAMY, K. M.; Quasi-projective abelian groups, *Bull. Soc. Math. France*, **98** (1970), 5–8.

[4] HAUSEN, J.; Supplemented modules over Dedekind domains, *Pacific J. Math.*

[5] KASCH, F.; "*Moduln and Ringe*," B. G. Teubner, Stuttgart, 1977.

[6] KASCH, F. and MARES, E.; Eine Kennzeichnung semi-perfekter Moduln, *Nagoya Math. J.*, **27** (1966), 525–529.

[7] MIYASHITA, Y.; Quasi-projective modules, perfect modules, and a theorem for modular lattices, *J. Fac. Sci. Hokkaido Univ.*, **19** (1966), 86–110.

[8] PIERCE, R. S.; Homomorphisms of primary abelian groups, Topics in Abelian Groups, Chicago, 1963, pp. 215–310.

[9] VARADARAJAN, K.; Modules with supplements, *Pacific J. Math.*, **82** (1979), 559–564.

[10] VASCONCELOS, W. V.; On finitely generated flat modules, *Trans. Amer. Math. Soc.*, **138** (1969), 505–512.

[11] WU, L. E. T. and JANS, J. P.; On quasi projectives, *Illinois J. Math.*, **11** (1967), 439–448.

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